CALCULATION OF DEVELOPED LAMINAR GAS FLOW IN A PLANE-PARALLEL CHANNEL

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A method is given for solving the Prandtl equations for adiabatic gas flow in a plane-parallel channel at Pr = 1. The method is based on reduction of the Prandtl equations to an approximate system of ordinary quasilinear, first-order differential equations and can be used to solve more complicated problems.

We shall consider adiabatic developed gas flow with Pr = 1 in a plane channel with parallel walls. We take as the velocity scale the limiting velocity; as the temperature scale, the stagnation temperature; as the pressure and density scales, the stagnation pressure and density at the center of the initial cross section; as the viscosity-coefficient scale, its value at the center of the initial cross section at the stagnation temperature; as the transverse-coordinate scale, one half the channel width (h); and as the longitudinal-coordinate scale the value $h\widetilde{\mathrm{Re}}$, where $\widetilde{\mathrm{Re}}$ is defined over h and the above-mentioned viscosity.

Below, all values are dimensionless. The prime indicates differentiation with respect to x. The system of equations in the Prandtl approximation is written as

$$\rho u \frac{\partial u}{\partial x} + \widetilde{\mathrm{Re}}\rho v \frac{\partial u}{\partial y} + \bar{k}p' - \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0, \quad (1)$$

$$\frac{\partial(\rho u)}{\partial x} + \widetilde{\mathrm{Re}}\frac{\partial(\rho v)}{\partial y} = 0, \quad (2)$$

$$p_1 = \rho(1-u^2), \quad (3)$$

$$\int_0^1 \rho u dy = G. \quad (4)$$

Here, $\bar{k} = (k-1)/2k$.

For the dependence of the viscosity on the temperature $T = 1 - u^2$, we take the power relation

$$\mu = (1-u^2)^{\beta}. \quad (5)$$

We approximate the longitudinal-velocity profile by the polynomial

$$u = \sum_{i=1}^{n} f_i(x) y^i, \quad (6)$$

where the coefficients $f_i(x)$ are unknown functions of x.

We shall require that profile (6) satisfy (1) near the wall.

We substitute $\widetilde{\mathrm{Re}}\rho v$ from (2),

$$\widetilde{\mathrm{Re}}\rho v = -\int_0^y \frac{\partial(\rho u)}{\partial x} dy,$$

and also ρ and μ from (3) and (5) into (1). Expanding the left side of the obtained equation into a series in y, we find

$$\varphi_0 + \varphi_1 y + \varphi_2 y^2 + \ldots = 0, \quad (7)$$

where φ_i is a function only of x.

If we let

$$\varphi_0 = \varphi_1 = \varphi_2 = \ldots = \varphi_{l-1} = 0, \quad (8)$$

Eq. (1) will be satisfied near the wall.

Now we require that profile (6) satisfy Eq. (1) near the channel axis. For this we represent (6) as

$$u = \sum_{i=1}^{n} F_i (1-y)^i. \quad (9)$$

Note that F_i can be expressed in terms of f_i. These expressions are easy to find from the condition of coincidence of polynomials (6) and (9):

$$F_0 = \sum_{i=1}^{n} f_i, \ \ldots,$$

$$F_k = (-1)^k \sum_{i=1}^{n} C_i^k f_i, \ \ldots, \ F_n = (-1)^n f_n. \quad (10)$$

Considering that $v|_{y=1} = 0$, from (2) we find

$$\widetilde{\mathrm{Re}}\rho v = \int_0^{1-y} \frac{\partial(\rho u)}{\partial x} d(1-y)$$

and, as in the previous case, we obtain

$$\Phi_0 + \Phi_1(1-y) + \Phi_2(1-y)^2 + \ldots = 0,$$

where Φ_i are functions only of x.

Letting

$$\Phi_0 = \Phi_1 = \Phi_2 = \ldots = \Phi_{m-1} = 0, \quad (11)$$

we satisfy (1) near the channel axis.

To $l + m$ equations (8) and (11) we join three others:

a) the condition of conservation of flow rate (4);

b) the momentum equation

$$\frac{d}{dx}\left(\int_0^1 \rho u^2 dy\right) + \bar{k}p' + f_1 = 0, \quad (12)$$

c) the relation $(\partial u/\partial y)_{y=1} = 0$, which follows from the condition of maximum velocity at the channel axis.

The obtained system of $l + m + 3$ equations makes it possible to find $n + 1$ unknowns $(p, f_1, f_2, \ldots, f_n)$ if $l + m = n - 2$. To find these unknowns, we must specify the initial conditions $p|_{x=0}$, $f_i|_{x=0}$.

The method can be extended to more complicated cases, for example, flows with heat transfer, flows in convergent channels, etc.

As an example, let us consider the solution of the following problem. Let there be a parabolic velocity distribution when x = 0:

$$u = U(2y - y^2). \qquad (13)$$

If the fluid was incompressible, then profile (13) would be retained over the entire extent of the flow,

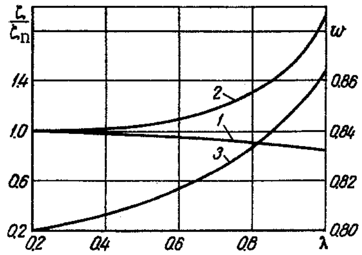

Fig. 1. Friction coefficient versus mean mass flow rate: 1) first approximation; 2) second approximation; 3) variation of the degree of filling of the velocity profile in the second approximation.

and U would remain constant, and for this reason the friction coefficient $\zeta = 8\tau_0/\widetilde{\mathrm{Re}}Gw$, where w is the mean mass flow rate, would also be constant. In the case of a gas, the profile must be deformed, and, because of this, the coefficient ζ must vary.

Let us consider two cases: a) we satisfy only integral relations (4) and (12) and the condition $(\partial u/\partial y)_{y=1} = 0$ (first approximation); and b) besides the three equations of a), we satisfy one relation near the wall and one near the axis $\varphi_0 = 0$ and $\Phi_0 = 0$ (second approximation).

In approximating polynomial (6) we should let n = 2 in case a) and n = 4 in case b).

The values φ_0 and Φ_0 are easily found from (1):

$$\varphi_0 = -\bar{k}p' - 2f_2,$$

$$\Phi_0 = \frac{pF_0}{1-F_0^2}\frac{dF_0}{dx} + \bar{k}p' - 2F_2(1-F_0^2)^\beta .$$

In the latter equation, F_0 and F_2 are expressed in terms of f_i according to (10).

Assuming that V in (13) is known, it is easy to find G according to (4) and (3):

$$G = \int_0^1 \rho u dy = (pI_1)_{x=0},$$

where

$$p|_{x=0} = (1-U^2)^{\frac{k}{k-1}}, \quad I_1 = \int_0^1 \frac{u}{1-u^2}dy.$$

In accordance with (13), the initial conditions for f_1 and f_2 in cases a) and b) are

$$f_1 = 2U, \quad f_2 = -U.$$

In case b), when x = 0 we must have $f_3 = f_4 = 0$.

Figures 1 and 2 show some results of calculations in which k = 1.4, β = 1, and U = 0.1. The reduced mean mass flow rate

$$\lambda = \sqrt{\frac{k+1}{k-1}\frac{\int_0^1 \rho u^2 dy}{\int_0^1 \rho u dy}} = \sqrt{\frac{k+1}{k-1}\frac{I_2}{I_1}},$$

where

$$I_2 = \int_0^1 \frac{u^2}{1-u^2}dy,$$

is plotted on the axis of the abscissas.

The friction coefficient ζ was calculated by the formula

$$\zeta = \frac{8f_1}{\widetilde{\mathrm{Re}}G}\frac{I_1}{I_2}$$

and adjusted to $\zeta|_{x=0}$.

The degree of filling of the velocity profile ω is given by

$$\omega = I_2/F_0 I_1.$$

In the first approximation, the degree of filling of the profile is held constant, unlike the second approximation, in which ω increases appreciably. The nature of variation of the friction coefficient also differs. The increase in ζ in the second approximation agrees with the known results of boundary-layer theory for external problems [1] (an increase in λ corresponds to an increase in the negative pressure gradient).

In the first approximation, the velocity profile u/u_1 is not deformed. The deformation of the velocity profile in the second approximation is shown in Fig. 2. As λ increases, the profile, having become more filled, moves closer to the wall, which increases the friction coefficient.

The results indicate that purely integral methods without contour relations should be used with caution, since these methods can sometimes lead to results that are far from true, both quantitatively and qualitatively.

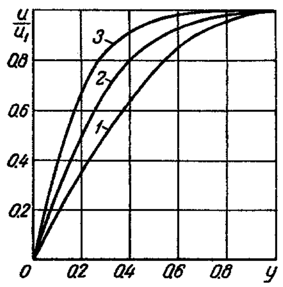

Fig. 2. Deformation of the velocity profile in the second approximation: 1) λ = 0.2; 2) λ = 0.69; 3) λ = 1.0.

The conclusions of this paper are in qualitative agreement with those of [2] for turbulent flows: as a critical region is approached, the velocity profile is filled all the more intensively.

NOTATION

x is the longitudinal coordinate; y is the transverse coordinate; v is the transverse velocity; u is the longitudinal velocity; p is the pressure; ρ is the density; μ is the dynamic viscosity; k is the isentropic exponent; G is the mass rate of gas discharge through half-channel width; β is the exponent in the power relation of the viscosity and temperature; λ is the reduced average mass rate; ω is the degree to which the velocity profile is filled; ζ is the friction coefficient.

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